MPP-Based Dimension Reduction Method for Accurate Prediction of the Probability of Failure of a Performance Function

Byungsu Kang[1], Chang-Eob Kim[2], Hyunkyoo Cho[3], K. K. Choi[3] and Dong-Hun Kim[1]

[1]Dept. of Electrical Eng., Kyungpook National Univ., Daegu 41566, Republic of Korea, dh29kim@ee.knu.ac.kr
[2]Dept. of Electrical Eng., Hoseo Univ., Chungnam 336-795, Republic of Korea
[3]Dept. of Mechanical and Industrial Eng., Univ. of Iowa, Iowa City, IA 52242-1527, USA

A hybrid reliability analysis method is proposed to yield very accurate failure rate calculation of a performance function when dealing with multi-dimensional and nonlinear electromagnetic (EM) systems in the present of uncertainties. To achieve this goal, the first-order reliability method (FORM) called reliability index approach (RIA) is first conducted for searching a most probable failure point (MPP) at a given design. However, the reliability analysis result using FORM may have significant errors especially for nonlinear and multi-dimensional functions. To overcome the drawback, the univariate dimension reduction method (DRM) is additionally executed at the obtained MPP, and then the probability of failure of a performance function is recalculated using an additive decomposition of *N*-dimensional function into *N* one-dimensional functions. Two test problems are provided to demonstrate numerical efficiency and accuracy of the proposed method by comparison with existing reliability methods.

***Index Terms*—Electromagnetics, optimization, reliability theory, robustness.**

I. Introduction

In resent years, there have been various attempts to develop reliability analysis methods which can accurately compute the probability of failure of a EM performance function in the present of uncertainties [1]-[5]. Among them, the MPP-based method such as FORM or second-order reliability method (SORM) is very popular since it can be very effectively used for reliability assessment [3]-[5]. However, the reliability analysis using FORM could be very erroneous because FORM cannot reflect complexity of nonlinear and multi-dimensional functions. Although the reliability analysis using SORM may be accurate, it is not easy to use for practical engineering problems. That is because SORM requires the second-order sensitivities which are very difficult and expensive to obtain. To alleviate these difficulties, this paper proposes a hybrid reliability analysis method where a univariate DRM is incorporated with FORM. The method can estimate the probability of failure of a performance function more accurately than FORM and more efficiently than SORM.

II. Proposed MPP-Based DRM

The statistical description of the failure of the performance/constraint function g is characterized by the cumulative distribution P_f as

$$\begin{aligned} &P[g(\mathbf{x}) > 0] = P_f \\ &P_f = \int_{g(\mathbf{x})>0} \cdots \int f_{\mathbf{x}}(\mathbf{x})\, dx_1 \cdots dx_n . \end{aligned} \tag{1}$$

In (1), **x** is the mean of random variables **x** in *X*-space, and $f_{\mathbf{x}}(\mathbf{x})$ is the joint probability density function of all random variables in n dimensional space. To handle the multiple integrations in (1), the first-order Taylor series in FORM are adopted to approximate the constraint function. Therein, a transformation T is needed from the original random variable **x** to the independent and standard normal random variable **u** in *U*-space as shown in Fig. 1[1], [5].

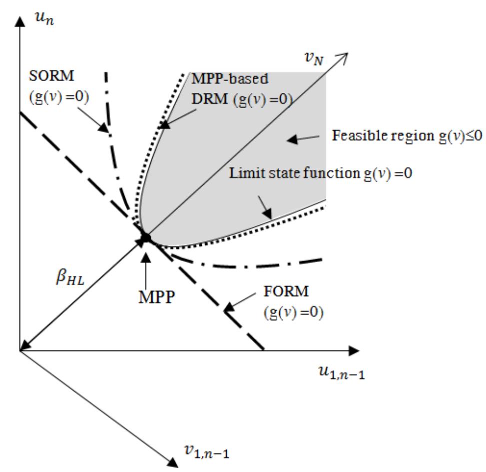

Fig. 1. Illustration of three different approximation functions of *g*.

A. Failure Rate Calculation using RIA

In RIA as one of FORMs, the MPP can be found by solving the following optimization:

$$\begin{aligned} &\text{minimize} \qquad \|\mathbf{u}\| \\ &\text{subject to} \qquad g(\mathbf{u}) = 0. \end{aligned} \tag{2}$$

After finding the MPP ($\mathbf{u}^*$), the Hasofer-Lind reliability index β_{HL} in Fig. 1 is obtained by measuring the distance between MPP and the origin in *U*-space. The probability of failure is approximated by using a linear approximation of g (referred to the dashed line in Fig. 1) as

$$P_f^{\mathrm{FORM}} \approx \Phi(-\beta_{HL}) \tag{3}$$

where $\Phi(\bullet)$ is the standard normal cumulative distribution function of the standard Gaussian random variable. However, the reliability assessment using (3) inherently includes a significant error due to the unsuitable approximation of a highly nonlinear function g as in Fig. 1.

B. Failure Rate Recalculation using DRM

To resolve the aforementioned difficult, the univariate DRM is carried out at the FORM-based MPP $\mathbf{u}^*$. Since the probability of failure cannot be directly calculated U-space, a rotated standard normal V-space is newly introduced as seen in Fig. 1, where $\mathbf{u}^*$ is defined by $\mathbf{v}^*=[0,\cdots,0,\beta_{HL}]^T$ in V-space. The n-dimensional performance function $g(\mathbf{u})$ is additively decomposed into one-dimension ones [3], [4]:

$$g(\mathbf{u}) \approx \hat{g}(\mathbf{v}) \equiv \sum_{i=1}^{n}\sum_{j=1}^{m} w_i^j g(v_1,\cdots,v_i^j,\cdots,v_n) - (n-1)g(\mathbf{v}^*) \tag{4}$$

where m is the number of weights and quadrature points, and the symbols, w_i^j and v_i^j, mean the jth weight factor and quadrature point for the ith random variable v_i, respectively. Based on Gaussian quadrature theory, m quadrature points and weights yield a degree of precision of $2m$-1 (referred to the dotted line in Fig. 1). Finally, the probability of failure using the MPP-based DRM is calculated as

$$P_f^{DRM} = \prod_{i=1}^{n}\sum_{j=1}^{m} w_j \Phi(-\beta_{HL} + \hat{g}_i(v_i^j)/b_1)/\Phi(-\beta_{HL})^{n-2} \tag{5}$$

where $\hat{g}_i(v_i) \equiv \hat{g}(0,\cdots 0, v_i, 0,\cdots,\beta_{HL})$ is a function of v_i only and $b_1=\|\partial g(\mathbf{u}^*)/\partial u_i\|$.

III. Results

Numerical accuracy and efficiency of the proposed method is verified by comparing it with the probability of failure obtained using existing methods. For this purpose, the Monte Carlo simulation (MCS) result is regarded as an exact one.

For the first example, a highly nonlinear fourth-order polynomial function of (6) is tested.

$$\begin{aligned} &g(y,z) = 0.7361 + (y-6)^2 + (y-6)^3 - 0.6(y-6)^4 + z \\ &y = 0.9063x_1 + 0.4226x_2, \quad z = 0.4226x_1 - 0.9063x_2 \end{aligned} \tag{6}$$

where two random variables, x_1 and x_2, are assumed to comply with normal probabilistic distributions of $N(4,0.3)$ and $N(3,0.3)$ for the failure rate computation, respectively. In Table I, four different reliability analysis methods were used: FORM, SORM, proposed MPP-based DRMs with three and five quadrature points and MCS. It is observed that the proposed method with five quadrature points is even more accurate and efficient than SORM when compared with the MCS result. Meanwhile, although FORM requires the smallest number of function calls for reliability assessment, its solution error reaches up to almost 52%.

TABLE I
Probability Of Failure Calculation By Four Different Methods For An Analytical Example

	FORM	SORM	Proposed 3 points	Proposed 5 points	MCS
P_f (%)	5.00	3.41	3.59	3.39	3.28
Function calls	14	25	16	18	300,000

For the second example, the TEAM benchmark problem 22 of a superconducting magnetic energy storage system (SMES) in Fig. 2 is considered. In [5], the original three-parameter problem with a design vector $\mathbf{d}=[R_2, D_2, H_2]^T$ was modified so as to be suitable for the reliability-based design optimization (RBDO). Three different SMES designs were presented therein: initial, deterministic design optimization (DDO) and RBDO points. Here, a performance constraint function of (7) is tested at each design point.

$$g(\mathbf{x}) = 1 - \left((E(\mathbf{x}) - E_o)/(0.05\times E_o)\right)^2 \tag{7}$$

where the components of a random vector $\mathbf{x}=[x_1, x_2, x_3]^T$ follow normal probabilistic distributions, of which standard deviation values are 10, 5 and 10, respectively. The symbol E is the stored magnetic energy with the target value E_o of 180 MJ.

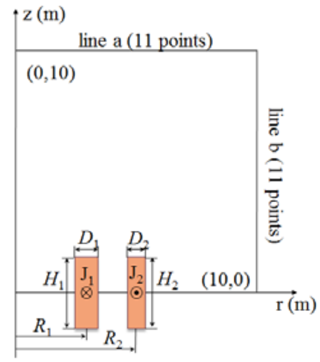

Fig. 2. Configuration of the SMES device.

The SMES model was discretized into 6,421 triangular elements with the second-order interpolation function, and EM simulations were executed with a commercial finite element analysis code, called MagNet [6]. In Table II, the failure rate evalutions of (7) were conducted by using three different reliability analysis methods: FORM, proposed MPP-based DRM with three quadrature points and MCS. With reference to MCS results, it is obvious that the proposed method yields more accurate solutions than FORM at all design points.

It is inferred that the proposed method can estimate the probability of failure of a performance function as accurately as SORM without requiring the second-order sensitivity calculation and much more accurately than FORM. More detailed explanation and comparative results will be presented in the extended version of the paper.

TABLE II
Probability Of Failure Calculation By Three Different Methods For A SMES Model

		FORM	Proposed	MCS
P_f (%)	Initial design, **d**= [2340,310,1780]	10.32	16.60	15.26
	DDO design, **d**= [2335,238,1853]	31.94	30.63	30.95
	RBDO design, **d**= [2347,234,1864]	5.00	5.64	5.84

Other six design variables were fixed as R_1=1,977 mm, D_1=404 mm, H_1=1,507 mm, J_1=16.30 A/mm^2, and J_2=16.19 A/mm^2 and MCS was performed with 500,000 samples.

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